The SLOPE inequality

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Notation. For $P(x) = \sum_{k\geq 0} p_k x^k$ with nonnegative coefficients, $p_k = [x^k]P$. A nonnegative sequence is *unimodal* if $p_0 \leq \cdots \leq p_m \geq p_{m+1} \geq \cdots$ for some m; $\mathrm{mode}(P) = \min\{k : p_k = \max_i p_i\}$ (left-most argmax). "$S \leq D$ coefficientwise" means $s_k \leq d_k$ for all k.

The inequality. Let D, S be nonnegative sequences with $S \leq D$, let $A := D + xS$ (so $a_k = d_k + s_{k-1}$), and for an integer $j \geq 1$ put $E := (1+x)^j A$, $M := \mathrm{mode}(E)$. Assuming $\mathrm{mode}(D) \leq M - 3$, consider

$$\textbf{(SLOPE)} \qquad e_{M-1} - e_{M-2} \;\geq\; d_{M-3} - d_{M-2}.$$

Proved case ($S \equiv 0$).

Theorem 1. *If $S \equiv 0$ (so $A = D$, $E = (1+x)^j D$), D is unimodal, $j \geq 4$ and $\mathrm{mode}(D) = M - 3$, then **(SLOPE)** holds. The same holds for every mode-drop $\mathrm{mode}(D) \leq M - 3$.*

(Proof and an explicit Farkas certificate: https://erdos993.shisheng.li/slope-pureD.pdf.)

The general case is independence-polynomial-specific. For $S \not\equiv 0$, **(SLOPE)** is the inequality that arises when

$$D = I(G \setminus w), \quad S = I(G \setminus N[w]), \quad A = I(G) = D + xS$$

are independence polynomials of a rooted forest (G, w); in that setting it has been verified extensively and is conjectured to hold. However, it is *not* a clean abstract inequality on the sequences: under each of the natural hypotheses "D unimodal", "D log-concave", "A log-concave" (with $S \leq D$, A unimodal), **(SLOPE)** has counterexamples. For instance

$$D = (6, 5, 4, 3), \quad S = (1, 2, 4, 0), \quad j = 1$$

has D log-concave and strictly decreasing, $A = (6, 6, 6, 7)$, $M = 3$, $\mathrm{mode}(D) = 0 = M - 3$, yet $e_2 - e_1 = 0 < 1 = d_0 - d_1$. (All such counterexamples have $\mathrm{mode}(D) = 0$, which an independence polynomial $I(G \setminus w)$ never has, since its constant term is 1.) Finding the correct hypotheses for the general case — or a direct proof for independence polynomials — is the open problem.

Problem 1. *Determine whether **(SLOPE)** holds for $D = I(G \setminus w)$, $S = I(G \setminus N[w])$, $A = I(G)$ of every rooted forest (G, w) and every $j \geq 1$ with $\mathrm{mode}(D) \leq M - 3$.*