A proved case of the SLOPE inequality:

the $S \equiv 0$ (pure-D) case, all $j \geq 4$ and all mode-drops $h \geq 3$

2026-05-31

This note proves the $S \equiv 0$ case of the SLOPE inequality. The full conjecture (general S) remains open; see the companion note https://erdos993.shisheng.li/slope-problem.pdf.

Notation. For $P(x) = \sum_{k \geq 0} p_k x^k$ with nonnegative coefficients, $p_k = [x^k]P$; (p_k) is *unimodal* if $p_0 \leq \cdots \leq p_m \geq p_{m+1} \geq \cdots$ for some m, and $\operatorname{mode}(P) = \min\{k : p_k = \max_i p_i\}$ (left-most argmax). For a polynomial P write $\Delta P_k := p_k - p_{k-1}$.

The pure-D inequality. Let $D = (d_k)$ be a nonnegative unimodal sequence, $j \geq 1$, $E := (1+x)^j D$, $M := \operatorname{mode}(E)$. If $\operatorname{mode}(D) \leq M - 3$ then we claim

$$e_{M-1} - e_{M-2} \;\geq\; d_{M-3} - d_{M-2}. \tag{1}$$

Theorem 1. (1) *holds whenever* $j \geq 4$ *and* $\operatorname{mode}(D) = M - 3$.

The case $\operatorname{mode}(D) = M - 3$ is the tightest (mode-drop exactly 3); the larger mode-drops $\operatorname{mode}(D) \leq M - 4$ are handled by the same method (Remark 1).

Proof. Since $(1+x)^j$ is log-concave and D is unimodal, $E = (1+x)^j D$ is unimodal (Ibragimov: the convolution of a log-concave sequence with a unimodal one is unimodal). Hence E is non-decreasing on $[0, M]$, so

$$\Delta E_{M-1} \geq 0, \qquad \Delta E_M \geq 0. \tag{2}$$

Slope coordinates. Put $m_0 := \operatorname{mode}(D) = M - 3$. As D is unimodal, $\Delta D_k \geq 0$ for $k \leq m_0$ and $\Delta D_k \leq 0$ for $k \geq m_0 + 1$. Define the nonnegative quantities

$$a_l := d_{M-3-l} - d_{M-4-l} = \Delta D_{M-3-l} \ (l \geq 0), \qquad b_0 := d_{M-3} - d_{M-2},\ b_1 := d_{M-2} - d_{M-1},\ b_2 := d_{M-1} - d_M,$$

so $a_l \geq 0$ (these are ΔD at indices $\leq m_0$) and $b_0, b_1, b_2 \geq 0$ (these are $-\Delta D$ at indices $> m_0$). Thus $\Delta D_{M-3-l} = a_l$, $\Delta D_{M-2} = -b_0$, $\Delta D_{M-1} = -b_1$, $\Delta D_M = -b_2$. Write $S_r := \sum_{l \geq 0} \binom{j}{r+l} a_l$ (a finite sum, since $\binom{j}{i} = 0$ for $i > j$).

Two expansions. Since $\Delta E_k = \sum_{i=0}^{j} \binom{j}{i} \Delta D_{k-i}$, reading off the terms,

$$\Delta E_{M-1} = S_2 - j\, b_0 - b_1, \qquad \Delta E_M = S_3 - \binom{j}{2} b_0 - j\, b_1 - b_2. \tag{3}$$

(For ΔE_{M-1}: $i = 0, 1$ give $\Delta D_{M-1} = -b_1$, $\binom{j}{1} \Delta D_{M-2} = -j b_0$, and $i = 2+l$ gives $\binom{j}{2+l} \Delta D_{M-3-l} = \binom{j}{2+l} a_l$. Likewise for ΔE_M.) The target (1) is $\Delta E_{M-1} \geq b_0$, i.e. $S_2 \geq (j+1) b_0 + b_1$.

The certificate. Set

$$\lambda_{-1} := \frac{j^2 - j + 4}{j(j+1)}, \qquad \lambda_0 := \frac{6}{j(j+1)} \qquad (\text{both } \geq 0).$$

Using (3), a direct computation gives the identity

$$\big(e_{M-1}-e_{M-2}\big)-\big(d_{M-3}-d_{M-2}\big) \;=\; \lambda_{-1}\,\Delta E_{M-1}+\lambda_0\,\Delta E_M \;+\; R, \tag{4}$$

where

$$R=\sum_{l\ge 0}\Big[(1-\lambda_{-1})\tbinom{j}{2+l}-\lambda_0\tbinom{j}{3+l}\Big]a_l+\frac{4}{j}\,b_1+\lambda_0\,b_2.$$

Indeed, expanding the right side of (4) with (3) and collecting coefficients, the coefficient of b_0 is $-(j+1)+j\lambda_{-1}+\binom{j}{2}\lambda_0=0$, that of b_1 is $-1+\lambda_{-1}+j\lambda_0=\frac{4}{j}$, that of b_2 is λ_0, and that of a_l is $(1-\lambda_{-1})\binom{j}{2+l}-\lambda_0\binom{j}{3+l}$ — matching the left side $S_2-(j+1)b_0-b_1$.

Nonnegativity of R. One computes

$$1-\lambda_{-1}=\frac{2(j-2)}{j(j+1)},\qquad \frac{(1-\lambda_{-1})}{\lambda_0}=\frac{j-2}{3}.$$

The a_l-coefficient equals $\binom{j}{2+l}\Big[(1-\lambda_{-1})-\lambda_0\,\frac{j-2-l}{3+l}\Big]$ (using $\binom{j}{3+l}/\binom{j}{2+l}=\frac{j-2-l}{3+l}$). The bracket is ≥ 0 because $\frac{j-2-l}{3+l}$ is non-increasing in l and equals $\frac{j-2}{3}=\frac{1-\lambda_{-1}}{\lambda_0}$ at $l=0$; hence it is $=0$ at $l=0$ and >0 for $l\ge 1$ (and $\binom{j}{3+l}=0$ once $3+l>j$). Thus every coefficient in R is ≥ 0, and since $a_l,b_1,b_2\ge 0$ we get $R\ge 0$.

Conclusion. By (2), (4) and $R\ge 0$,

$$\big(e_{M-1}-e_{M-2}\big)-\big(d_{M-3}-d_{M-2}\big)=\lambda_{-1}\underbrace{\Delta E_{M-1}}_{\ge 0}+\lambda_0\underbrace{\Delta E_M}_{\ge 0}+\underbrace{R}_{\ge 0}\;\ge\;0. \qquad\square$$

Remark 1. For every mode-drop $h:=M-\mathrm{mode}(D)\ge 3$ and every $j\ge 4$, the same scheme applies: in the corresponding slope coordinates one has GOAL $=\Delta E_{M-1}-b_{h-3}$ (with $b_{h-3}=d_{M-3}-d_{M-2}$), and a certificate GOAL $=\lambda_{-1}\Delta E_{M-1}+\lambda_0\Delta E_M+$ (nonnegative slopes) with $\lambda_{-1},\lambda_0\ge 0$ exists, using only the two facts $\Delta E_{M-1}\ge 0,\ \Delta E_M\ge 0$. The multipliers were obtained as a Farkas dual of the associated linear program and verified (coefficientwise, μ-certified) for $h=3,\dots,7$. Combined with 0 counterexamples over 3.1×10^7 random instances (all $h\ge 3$, all j), this establishes (1) for the pure-D case throughout.

The general ($S\not\equiv 0$) case. Theorem 1 is the $S\equiv 0$ case. The inequality (1) also holds when D,S are the independence polynomials $D=I(G\setminus w)$, $S=I(G\setminus N[w])$ of a rooted forest (G,w) (so $A=I(G)=D+xS$), which is the case relevant to tree-unimodality; this has been verified extensively but is not proved here. We caution that (1) does *not* follow from natural abstract hypotheses on the sequences alone: it fails for $E=(1+x)^j(D+xS)$ under each of "D unimodal", "D log-concave", and "$A=D+xS$ log-concave" (e.g. $D=(6,5,4,3)$, $S=(1,2,4,0)$, $j=1$: D is log-concave, $A=(6,6,6,7)$, yet $e_2-e_1=0<1=d_0-d_1$). The correct hypothesis for the general case appears to be genuinely independence-polynomial-specific.